Phase 10 – Part 2  
Dispersion Relations for ψ Perturbations  
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Goal  
The purpose of this part is to derive explicit dispersion relations for small perturbations φ(x, t) on top of stationary ψ wells. Having linearized the governing dynamics in Part 1, I now impose a wave-like ansatz for perturbations and compute ω–k relations, which describe how frequency depends on wavenumber in ψ-gravity. These relations reveal whether perturbations propagate stably, oscillate, or grow unstably.

Recap of Linearized Framework  
From Part 1:

Plain text:  
ψ(x, t) = ψ₀(x) + ε φ(x, t)

The linearized force in a homogeneous region (constant curvature ) was:

Plain text:  
Force(x, t) ≈ −ε C₀ ∇φ(x, t)

Effective Perturbation Dynamics  
In analogy to Newtonian particle dynamics, acceleration is proportional to force. The perturbation field obeys a wave-like equation (taking ε normalized into the perturbation amplitude):

Plain text:  
∂²φ/∂t² = −C₀ ∇²φ(x, t)

This is a wave equation with stiffness parameter .

Wave Ansatz  
Insert the standard plane-wave solution:

Plain text:  
φ(x, t) = exp(i (k·x − ω t))

Compute derivatives:

Temporal:

Plain text:  
∂²φ/∂t² = −ω² φ

Spatial:

Plain text:  
∇²φ = −k² φ

Dispersion Relation  
Substitute into the wave equation:

which simplifies to

Plain text:  
ω² = C₀ k²

This is the dispersion relation for ψ perturbations.

Interpretation

* If :
* Plain text:  
  ω = ±√(C₀) k
* This corresponds to stable oscillatory wave propagation, with phase velocity:
* Plain text:  
  vₚ = √(C₀)
* and group velocity:
* Plain text:  
  v\_g = √(C₀)
* Thus waves are non-dispersive in homogeneous ψ regions.
* If :
* Plain text:  
  ω² = −|C₀| k²
* Here, ω is purely imaginary:
* Plain text:  
  ω = ±i √(|C₀|) k
* This corresponds to exponential growth or decay of perturbations — an instability regime.
* If :
* Plain text:  
  ω = 0
* Perturbations are static, corresponding to neutral equilibrium.

Desert Analogy Translation

* Positive curvature : ripples on the desert floor oscillate like sound waves in the sand–wind medium.
* Negative curvature : ripples collapse into dunes that grow exponentially, destabilizing the surface.
* Zero curvature : the desert floor is flat and disturbances neither grow nor propagate.

Numerical Example (Dispersion Curve)

# simulations/phase10\_part2\_dispersion.py  
import numpy as np  
import matplotlib.pyplot as plt  
  
# Define curvature values  
C\_positive = 1.0  
C\_negative = -1.0  
  
# Wavenumber range  
k = np.linspace(0, 5, 200)  
  
# Dispersion relations  
omega\_pos = np.sqrt(C\_positive) \* k  
omega\_neg = 1j \* np.sqrt(abs(C\_negative)) \* k # imaginary for instability  
  
plt.plot(k, omega\_pos, label="Stable (C0 > 0)")  
plt.plot(k, np.imag(omega\_neg), '--', label="Unstable growth rate (C0 < 0)")  
plt.xlabel("Wavenumber k")  
plt.ylabel("Frequency ω")  
plt.title("Phase 10 Part 2: Dispersion Relations in ψ-Gravity")  
plt.legend()  
plt.grid(True)  
plt.show()